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A Subspace Inclusion Graph of a Finite Dimensional Vector Space

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Abstract: The combination of algebraic structures and graphs was carried out by investigating thoroughly the relation among the algebraic structure and the graph theoretic properties. Moreover, it needs to explore algebraic structure. The results of the combination of algebraic structures and graphs have many applications in the fields of Internet modeling, coding, etc**.** For example, the famous Cayley graph was constructed from groups and widely used in network models. Das introduced the subspace inclusion graph on finite-dimensional vector space over a finite filed, and studied that the graph is bipartite and some special properties if the dimension of the vector space is 3. In this paper, this bipartite inclusion graph in the case of 3-dimensional is extended to more general dimensional bipartite inclusion graph. The diameter, girth, clique number, covering number, independence number and matching number are studied and the properties are shown, such as regular, planar and Eulerian. Moreover, the authors also introduce a new results about the structure and the number of 1-dimensional and *n*-1-dimensional subspaces on *n*-dimensional vector space.

Index Terms: Vector space, Subspace, Inclusion graph.
Mathematics Subject Classification: 05C25, 05C69, 20B25.

1. Introduction

In [1], Das defined a subspace inclusion graph $\mathscr{I}n(\mathbb{V})$ on *n*-dimensional vector space $\mathbb{V}$ over a finite field F_q with q elements as follows: the vertex set V of $\mathscr{I}n(\mathbb{V})$ is the collection of nontrivial proper subspaces of $\mathbb{V}$ and for $W_1, W_2 \in V, W_1$ is adjacent to W_2 if either $W_1 \subset W_2$, or $W_2 \subset W_1$. Das had shown that $\mathscr{I}n(\mathbb{V})$ is bipartite and some graph theoretic properties of $\mathscr{I}n(\mathbb{V})$, if the dimension *n* of $\mathscr{I}n(\mathbb{V})$ is 3. Note that, for *n*=3, the vertex set of $\mathscr{I}n(\mathbb{V})$ is the union of 1-dimensional subspaces and 2-dimensional subspaces. Obviously, the scope of application of graph $\mathscr{I}n(\mathbb{V})$ is very limited. In this paper, the bipartite graph in the case *n*=3 be extended to more general dimensional bipartite inclusion graph, denoted by $\mathscr{I}nb(\mathbb{V})$, the vertex set *V* of $\mathscr{I}nb(\mathbb{V})$ is the collection of 1-dimensional subspaces and *n*-1-dimensional subspaces of *n*-dimensional vector space and denoted by W, U respectively. *W* is adjacent to *U* if either $W \subset U$. The diameter, girth, clique number, covering number, independence number and matching number are studied and we also show that the properties of $\mathscr{I}nb(\mathbb{V})$, such as regular, planar and Eulerian. More specifically, the authors also show that the structure and the number of 1-dimensional and *n*-1-dimensional subspaces on *n*-dimensional vector space.

2. Literature Review

The association graph with algebraic structure is an interesting and emerging topic in the research of algebraic graph theory. Much works has been done on these graphs with algebraic structure, such as with ring, group, vector space and so on. In [2], Beck introduced the idea of coloring of a commutative ring, and discussed the rings which are finitely colorable, leaving aside, for the moment, possible applications to graph theory. In 1999, Anderson and Livingston [3] studied the zero-divisor graph of a commutative ring. After that, in [4], Redmond studied the structure in the zero-divisor graph of a noncommutative ring. It showed that the zero-divisor graph of a finite ring which no nontrivial nilpotent elements was not a tournament and the graph has more than one vertex. This result is generalized to an arbitrary ring. The association graphs on group can be found in [5], it showed that the planarity and fixing number of inclusion graph of a nilpotent group.

In recent years, the association graph with vector space got a lot of attention. A subspace intersection graph on vector space has been studied in [6,7]. And obtained necessary and sufficient condition such that these intersection graphs are complete and characterize the vector spaces for which the graphs are isomorphic in [7]. A non-zero component graph theoretic properties with vector space were studied, such as the graph was connected and found its domination number and independence number in [8]. In [9], the sum graph on a vector space was introduced. And in [10], Das introduced the inclusion graph on vector space, the diameter, girth, clique number and chromatic number are studied. In this paper, we will introduce an inclusion graph $\mathcal{I}nb(\mathbb{V})$ with vector space.

3. Research Method

All the necessary definitions and the most basic properties are presented in Section 4. The authors introduce the graph theoretic properties, and in which, in order to calculate the covering number, independence number and matching number the authors give the number and structure of 1-dimensional and *n*-1-dimensional subspaces of the *n*-dimensional space in Section 5.

4. Definition and Preliminaries

All graphs considered in this paper are finite, simple and undirected, some concepts and symbols not mentioned in graph theory which will be used in the whole paper, see [11]. For a graph $\Gamma=(V(\Gamma),E(\Gamma))$ with the vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, we denote an edge of Γ by (u,v) with $u,v\in V(\Gamma)$. And we denote the number of vertices and edges in Γ, by $|V(\Gamma)|$ and $|E(\Gamma)|$. The *degree* of a vertex *v* in a graph Γ, denoted by *d(v)*, is the number of edges of Γ incident with *v*. For a vertex *v* of Γ, its *neighborhood*, denoted by *N(v)*, is the set of all vertices adjacent to *v*, and for two adjacent vertices *u* and *v* we write *u-v*. The length of a shortest path is called the *distance* between *x* and y and denoted by *d(x,y)*. The *diameter* of a graph Γ is the greatest distance between two vertices of Γ, and denoted by *diam*(Γ). The *girth* of a graph Γ is the length of a shortest cycle, denoted by $g(\Gamma)$. Specifically, if Γ doesn't contain any cycles, then $g(\Gamma)=\infty$. The *clique number* of graph Γ is the maximum size of a clique in Γ, denoted by $\omega(\Gamma)$. The cardinality of a minimum covering set of Γ is called the *covering number* of Γ and is denoted by $\tau(\Gamma)$. The cardinality of a maximum independent set of Γ is called the *independence number* of Γ and is denoted by $\pi(\Gamma)$. The maximum number of edges in a matching of a graph Γ is called the *matching number* of Γ and denoted by $\varepsilon(\Gamma)$. For a prime *p*, by F_p we denote a finite field about integers modulo *p* with addition and multiplication. For a prime power $q=p^n$, by F_q we denote the extension field of F_p. Let F_q^* be the multiplicative group of all the nonzero elements in F_q.

Let $\mathbb{V}(n,q)$ be a *n*-dimensional vector space over finite field F_q, the number of *k*-dimensional subspaces in $\mathbb{V}(n,q)$ is:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}. \tag{1}$$

Das defined a subspace inclusion graph on vector space in [1], we will define an inclusion subgraph about it as follows:

Definition 4.1 *Let* $\mathbb{V}(n,q)$ *be a n-dimensional vector space over finite field* F_q*. The inclusion graph* $\mathcal{I}nb(\mathbb{V})$ *with* $\mathbb{V}(n,q)$ *is defined as follows*:

$$V(\mathcal{I}nb(\mathbb{V}))=W(\mathcal{I}nb(\mathbb{V}))\bigcup U(\mathcal{I}nb(\mathbb{V})), \tag{2}$$

$$E(\mathcal{I}nb(\mathbb{V}))=\left\{(w,u)\mid w\in W(\mathcal{I}nb(\mathbb{V})), u\in U(\mathcal{I}nb(\mathbb{V})), w\subset u.\right\}. \tag{3}$$

Where $W(\mathcal{I}nb(\mathbb{V}))=\{1\text{-}dimensional\ subspaces\ of\ \mathbb{V}(n,q)\}$, $U(\mathcal{I}nb(\mathbb{V}))=\{n\text{-}1\text{-}dimensional\ subspaces\ of\ \mathbb{V}(n,q)\}$.

Note that there is no inclusion relationship between subspaces of the same dimension. According to the above definition, there is no edge in $W(\mathcal{I}nb(\mathbb{V}))$ and $U(\mathcal{I}nb(\mathbb{V}))$. Thus, we can get the following theorem:

Theorem 4.2 $\mathcal{I}nb(\mathbb{V})$ *is a bipartite graph.*

In the following, we call the graph $\mathcal{I}nb(\mathbb{V})$ in Definition 4.1 bipartite inclusion graph.

Theorem 4.3 *Let* $\mathbb{V}(n,q)$ *be a n-dimensional vector space over finite field* F_q*, then*

$$\left|V(\mathcal{I}nb(\mathbb{V}))\right| = 2\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{2(q^n - 1)}{q - 1}, \tag{4}$$

and for any $v \in V(\mathcal{I}nb(\mathbb{V}))$,

$$d(v) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q = \frac{q^{n-1} - 1}{q - 1}. \tag{5}$$

Especially, $\mathcal{I}nb(\mathbb{V})$ *is regular*.

Proof By equation (1), we can get the number of 1-dimensional and n-1-dimensional subspaces of n-dimensional vector space $\mathbb{V}(n,q)$ over finite field F_q is $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q$, respectively. Thus

$$\left|V(\mathcal{I}nb(\mathbb{V}))\right| = \left|W(\mathcal{I}nb(\mathbb{V}))\right| + \left|U(\mathcal{I}nb(\mathbb{V}))\right| = \begin{bmatrix} n \\ 1 \end{bmatrix}_q + \begin{bmatrix} n \\ n-1 \end{bmatrix}_q = 2\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{2(q^n - 1)}{q - 1}. \tag{6}$$

For any $u \in U(\mathcal{I}nb(\mathbb{V}))$, the degree of u is the number of 1-dimensional subspaces contained in the n-1-dimensional subspaces on the vector space $\mathbb{V}(n,q)$ over finite field F_q, then

$$d(u) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = \frac{q^{n-1} - 1}{q - 1}. \tag{7}$$

For any $w \in W(\mathcal{I}nb(\mathbb{V}))$, we know that the $d(w)$ is the number of n-1-dimensional subspaces containing given 1-dimensional subspaces. Since for any two 1-dimensional subspaces, the number of n-1-dimensional subspaces containing these two 1-dimensional subspaces is the same. By inclusion relationship and edge set, we know that $\left|W(\mathcal{I}nb(\mathbb{V}))\right| d(w) = \left|U(\mathcal{I}nb(\mathbb{V}))\right| d(u)$. Then we can get that $d(w)=d(u)$. Thus, for any $v \in V(\mathcal{I}nb(\mathbb{V}))$,

$$d(v) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = \frac{q^{n-1} - 1}{q - 1}. \tag{8}$$

Especially, $\mathcal{I}nb(\mathbb{V})$ is regular.

Corollary 4.4 *If* $dim(\mathbb{V}(n,q)) = 2$, *then* $\mathcal{I}nb(\mathbb{V})$ *is an empty graph*.

In the following, unless otherwise specified, $\mathcal{I}nb(\mathbb{V})$ is bipartite inclusion graph with $\mathbb{V}(n,q)$, where $dim(\mathbb{V}(n,q)) \geq 3$.

Theorem 4.5 $\mathcal{I}nb(\mathbb{V})$ *is connected*.

Proof By Theorem 4.2, we need to prove that for any $w \in W(\mathcal{I}nb(\mathbb{V}))$, and $u \in U(\mathcal{I}nb(\mathbb{V}))$, there exists a path from w to u. Without loss of generality, we can assume that $w = \langle \alpha \rangle$ and $u = \langle \beta_1, \beta_2, \cdots, \beta_{n-1} \rangle$.

If $\alpha \in \langle \beta_1, \beta_2, \cdots, \beta_{n-1} \rangle$, then $(w,u) \in E(\mathcal{I}nb(\mathbb{V}))$. Therefore, there exists a path from w to u. If $\alpha \notin \langle \beta_1, \beta_2, \cdots, \beta_{n-1} \rangle$, this infer that $\alpha, \beta_1, \beta_2, \cdots, \beta_{n-1}$ are linearly independent.

Let $u_1 = \langle \alpha, \beta_1, \beta_2, \cdots, \beta_{n-2} \rangle \in U(\mathcal{I}nb(\mathbb{V}))$ and $w_1 = \langle \beta_1 \rangle \in W(\mathcal{I}nb(\mathbb{V}))$. Clearly, $w, w_1 \in N(u_1)$. Note that $\langle \beta \rangle \subset \langle \beta_1, \beta_2, \cdots, \beta_{n-1} \rangle$. Then $(w_1, u) \in E(\mathcal{I}nb(\mathbb{V}))$. Then there exists a path $w - u_1 - w_1 - u$. Since w and u are chosen arbitrarily, it follows that $\mathcal{I}nb(\mathbb{V})$ is connected.

5. The graph theoretic properties of graph $\mathcal{I}nb(\mathbb{V})$

In this section, we shall focus on the graph theoretic properties of graph $\mathcal{I}nb(\mathbb{V})$, for example, we calculate diameter, girth, clique number, covering number, independence number and matching number of graph $\mathcal{I}nb(\mathbb{V})$, studying the planar, Eulerian of graph $\mathcal{I}nb(\mathbb{V})$.

Theorem 5.1 *Let $\mathbb{V}(n,q)$ be a n-dimensional vector space over finite field F_q. Then $diam(\mathcal{I}nb(\mathbb{V}))=3$.*

Proof For any $v_1,v_2 \in V(\mathcal{I}nb(\mathbb{V}))$, the following three cases are discussed.

(i) Suppose that $v_1,v_2 \in W(\mathcal{I}nb(\mathbb{V}))$, Without loss of generality, suppose that $v_1=\langle\alpha\rangle, v_2=\langle\beta\rangle$. We can get α and β are linearly independent due to $v_1 \neq v_2$. There exist the vectors $\gamma_1,\gamma_2,\cdots,\gamma_{n-3}$ such that the vector $\alpha,\beta,\gamma_1,\gamma_2,\cdots,\gamma_{n-3}$ span a n-1-dimensional subspace. Let $u=\langle\alpha,\beta,\gamma_1,\gamma_2,\cdots,\gamma_{n-3}\rangle$ and $u\in U(\mathcal{I}nb(\mathbb{V}))$. Then v_1-u-v_2 is a path of length 2 from v_1 to v_2. Hence, $d(v_1,v_2)=2$.

(ii) Suppose that $v_1,v_2 \in U(\mathcal{I}nb(\mathbb{V}))$. Without loss of generality, we can suppose that $v_1=\langle\alpha_1,\alpha_2,\cdots,\alpha_{n-1}\rangle$, $v_2=\langle\beta_1,\beta_2,\cdots,\beta_{n-1}\rangle$. Then there must be the non-zero vector $\alpha\in v_1\bigcap v_2$. Let $w=\langle\alpha\rangle$, we can get a path from v_1 to v_2: v_1-w-v_2 where $w=\langle\alpha\rangle$. Hence, $d(v_1,v_2)=2$.

(iii) Suppose that $v_1\in W(\mathcal{I}nb(\mathbb{V}))$ and $v_2\in U(\mathcal{I}nb(\mathbb{V}))$. The proof is the same as Theorem 4.5, there exists a path of length 3. Hence, $d(v_1,v_2)=3$.

Then $diam(\mathcal{I}nb(\mathbb{V}))=3$.

Theorem 5.2 *The girth of $\mathcal{I}nb(\mathbb{V})$ is*:

$$g(\mathcal{I}nb(\mathbb{V}))=\begin{cases}\infty, & n=2;\\ 6, & n=3;\\ 4, & n>3.\end{cases} \tag{9}$$

Proof According to Corollary 4.4, $\mathcal{I}nb(\mathbb{V})$ is an empty graph, if n=2. Thus $g(\mathcal{I}nb(\mathbb{V}))=\infty$. In [12], Das proved $g(\mathcal{I}nb(\mathbb{V}))=6$ if n=3. In the following, we only need to consider that n>3. Let $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be a basis of $\mathbb{V}(n,q)$. Choose that $v_1=\langle\alpha_1\rangle, v_2=\langle\alpha_1,\alpha_2,\cdots,\alpha_{n-1}\rangle, v_3=\langle\alpha_2\rangle, v_4=\langle\alpha_1,\alpha_2,\ \cdots,\alpha_{n-3},\alpha_{n-2},\alpha_n\rangle$, then $v_1-v_2-v_3-v_4-v_1$ is a 4-cycle. Then $g(\mathcal{I}nb(\mathbb{V}))=4$ if n>3.

Theorem 5.3 *Let $\mathbb{V}(n,q)$ be a n-dimensional vector space over finite field F_q, then $\omega(\mathcal{I}nb(\mathbb{V}))=2$.*

Proof According to Theorem 4.2, $\mathcal{I}nb(\mathbb{V})$ is a bipartite graph. Note that there is no edge between two vertices in the same bipart. Then the maximal complete subgraph of $\mathcal{I}nb(\mathbb{V})$ is an edge in $\mathcal{I}nb(\mathbb{V})$. Then $\omega(\mathcal{I}nb(\mathbb{V}))=2$.

Theorem 5.4 *Let $\mathbb{V}(n,q)$ be a n-dimensional vector space over finite field F_q. If $dim(\mathbb{V}(n,q))\geq 4$, then graph $\mathcal{I}nb(\mathbb{V})$ is not planar.*

Proof Let $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be a basis of $\mathbb{V}(n,q)$, and $w_1,w_2,w_3\in W(\mathcal{I}nb(\mathbb{V}))$, $u_1,u_2,u_3\in U(\mathcal{I}nb(\mathbb{V}))$. Without loss of generality, we can assume that $w_1=\langle\alpha_1\rangle, w_2=\langle\alpha_2\rangle, w_3=\langle\alpha_1+\alpha_2\rangle, u_1=\langle\alpha_1,\alpha_2,\cdots,\alpha_{n-1}\rangle$, $u_2=\langle\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\alpha_n\rangle, u_3=\langle\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\alpha_{n-1}+\alpha_n\rangle$. Then we can get a complete bipartite subgraph X of graph $\mathcal{I}nb(\mathbb{V})$, as follows:

$$V(X)=\{w_1,w_2,w_3\}\bigcup\{u_1,u_2,u_3\}, \tag{10}$$

$$E(X)=\{(w_1,u_1),(w_1,u_2),(w_1,u_3),(w_2,u_1),(w_2,u_2),(w_2,u_3),(w_3,u_1),(w_3,u_2),(w_3,u_3)\}. \tag{11}$$

Clearly, we can get $X=K_{3,3}$, This infer that $\mathcal{I}nb(\mathbb{V})$ contains $K_{3,3}$. According to Kuratowski's Theorem, we get that $\mathcal{I}nb(\mathbb{V})$ is not planar.

Theorem 5.5 *Let $\mathbb{V}(n,q)$ be a n-dimensional vector space over finite field F_q, if n and q are both odd, then graph $\mathcal{I}nb(\mathbb{V})$ is Eulerian.*

Proof By Theorem 4.3, for any $v\in V(\mathcal{I}nb(\mathbb{V}))$,

$$d(v)=\begin{bmatrix} n-1 \\ 1 \end{bmatrix}=\frac{q^{n-1}-1}{q-1}. \tag{12}$$

Then $d(v)$ is even if n and q are both odd. Thus graph $\mathcal{I}nb(\mathbb{V})$ is Eulerian if n and q are both odd.

Before proving the next theorem, first we give the structure and the number of 1-dimensional and n-1-dimensional subspaces of $\mathcal{I}nb(\mathbb{V})$.

Let $\mathbb{V}(n,q)$ be a n-dimensional vector space over finite field F_q, and $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be a basis of $\mathbb{V}(n,q)$. Then the structure and the number of 1-dimensional and n-1-dimensional subspaces as follows:

1-dimensional subspaces:

$$\langle \alpha_i + a_{i+1}\alpha_{i+1} + a_{i+2}\alpha_{i+2} + \cdots + a_n\alpha_n \rangle \tag{13}$$

Where $a_{i+1},a_{i+2},\cdots,a_n \in F_q$. For given i, we can get the number of 1-dimensional subspace is q^{n-i}. Note that $i=1,2,\cdots,n,$ then the number of all of 1-dimensional subspace is $q^{n-1}+q^{n-2}+\cdots+1.$ This sum is equal $\frac{q^n-1}{q-1}.$ This infer 1-dimensional subspace is the form as (13).

n-1-dimensional subspaces:

$$\langle \alpha_{k_1},\alpha_{k_2},\cdots,\alpha_{k_{n-1}} \rangle \tag{14}$$

$$\langle \alpha_{i_1^1},\alpha_{i_1^2},\cdots,\alpha_{i_1^s},\alpha_{i_2^1}+a_1\alpha_{i_2^2},\alpha_{i_2^1}+a_2\alpha_{i_2^3},\cdots,\alpha_{i_2^1}+a_{n-s+1}\alpha_{i_2^{n-s}} \rangle \tag{15}$$

In case (14), $\alpha_{k_1},\alpha_{k_2},\cdots,\alpha_{k_{n-1}}$ are that n-1 different elements randomly selected from $\alpha_1,\ \alpha_2,\cdots,\alpha_n$, thus the number of n-1-dimensional subspaces in case (14) is C_n^{n-1}.

In case (15), $a_1,a_2,\cdots,a_{n-s-1}\in F_q^{\ *}$. And the construction of case (15) is performed in two steps, where $\alpha_{i_1^1},\alpha_{i_1^2},\cdots,\alpha_{i_1^s}$ are randomly selected s different elements from $\alpha_1,\alpha_2,\cdots,\alpha_n$ in the first step, and $\alpha_{i_2^1},\alpha_{i_2^2},\alpha_{i_2^3},\cdots,\alpha_{i_2^{n-s}}$ are the n-s different elements left after the first step randomly selected from $\alpha_1,\alpha_2,\cdots,\alpha_n$ in the second step, here $s=0,1,2,\cdots,n-2.$ Note that for given s, if $\alpha_{i_1^1},\alpha_{i_1^2},\cdots,\alpha_{i_1^s}$ chosen in each first step is different, the n-1-dimensional subspaces corresponding to it is also different. Thus, for given s, we can get the number of n-1-dimensional subspaces is $C_n^s(q-1)^{n-s-1}$.

Then the number of all of n-1-dimensional subspaces is $C_n^0(q-1)^{n-1}+C_n^1(q-1)^{n-2}+\cdots+C_n^{n-2}(q-1)+C_n^{n-1}$. This sum is equal $\frac{q^n-1}{q-1}$. This infer n-1-dimensional subspace is the form as case (14) and case (15).

Theorem 5.6 *Let* $\mathbb{V}(n,q)$ *be a n-dimensional vector space over finite field* F_q*, then*

$$\pi(\mathcal{I}nb(\mathbb{V}))=\tau(\mathcal{I}nb(\mathbb{V}))=\varepsilon(\mathcal{I}nb(\mathbb{V}))=\begin{bmatrix} n \\ 1 \end{bmatrix}_q=\frac{q^n-1}{q-1}. \qquad n,q\geq 3. \tag{16}$$

Proof Let $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be a basis of $\mathbb{V}(n,q)$. Since $\mathcal{I}nb(\mathbb{V})$ is a bipartite graph, we know that $\mathcal{I}nb(\mathbb{V})$ must have a perfect matching if $n,q\geq 3.$ Consider the vertices of $\mathcal{I}nb(\mathbb{V})$ by the above argument. We give the perfect matching which consists of two types of edges:

(i) $\langle \alpha_k \rangle - \langle \alpha_k,\alpha_{k+1},\cdots,\alpha_{k+n-2} \rangle$, where $k=1,2,\cdots,n,$ if $k>n,$ then $\alpha_k=\alpha_{k-n}$.

(ii) $\langle \alpha_{j_1}+a_1\alpha_{j_2}+\cdots+a_{s-2}\alpha_{j_{s-1}}+a_{s-1}\alpha_{j_s} \rangle - \langle \alpha_{i_1^1},\alpha_{i_1^2},\cdots,\alpha_{i_1^{n-s}},\alpha_{i_2^1}+b_1\alpha_{i_2^2},\alpha_{i_2^1}+b_2\alpha_{i_2^3},\cdots,\alpha_{i_2^1}+b_{s-1}\alpha_{i_2^s} \rangle.$

Where $a_1,a_2,\cdots,a_{s-1},b_1,b_2,b_{s-1}\in F_q^{\ *},\ \alpha_{j_1},\alpha_{j_2},\cdots,\alpha_{j_{s-1}},\alpha_{j_s}$ are the s different elements randomly selected from $\alpha_1,\alpha_2,\cdots,\ \alpha_n$ and the construction method of $\langle \alpha_{i_1^1},\alpha_{i_1^2},\cdots,\alpha_{i_1^{n-s}},\alpha_{i_2^1}+b_1\alpha_{i_2^2},\alpha_{i_2^1}+b_2\ \alpha_{i_2^3},\cdots,\alpha_{i_2^1}+b_{s-1}\alpha_{i_2^s} \rangle$ is same as the above case (15), where $s=2,3,\cdots,n.$

These edge sets are a perfect matching for graph $\mathcal{I}nb(\mathbb{V})$, then

$$\pi(\mathcal{I}nb(\mathbb{V})) = \tau(\mathcal{I}nb(\mathbb{V})) = \varepsilon(\mathcal{I}nb(\mathbb{V})) = \begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n - 1}{q - 1}. \qquad n, q \geq 3. \tag{17}$$

6. Conclusion

Since the bipartite inclusion graph $\mathcal{I}n(\mathbb{V})$ defined by Das is very restrictive, this paper extended to more general dimensional bipartite inclusion graph of a finite dimensional vector space, denoted by $\mathcal{I}nb(\mathbb{V})$ .The main goal was to show the basic properties of connectedness, girth, diameter, clique number, regular, planar and Eulerian. Moreover, such as covering number, independence number and matching number. As a topic of further research, one can know that the structure and the number of 1-dimensional and *n*-1-dimensional subspaces on *n*-dimensional vector space over a finite field. In the future, one can study the symmetry of $\mathcal{I}nb(\mathbb{V})$ and the property of distance regular of $\mathcal{I}nb(\mathbb{V})$.

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